

Absence of singular continuous spectrum for some geometric Laplacians

Leonardo A. Cano García

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Abstract

We provide two examples of spectral analysis techniques of Schrödinger operators applied to geometric Laplacians. In particular we show how to adapt the method of analytic dilation to Laplacians on complete manifolds with corners of codimension 2 finding the absence of singular continuous spectrum for these operators, a description of the behavior of its pure point spectrum in terms of the underlying geometry, and a theory of quantum resonances.

Introduction

Spectral geometry studies the interactions between the geometry of a Riemannian manifold and the spectral analysis of its associated Laplacian. These interactions have been deeply studied in the case in which the manifold is closed (see [Cha84] [Ros97]). In the closed case the Laplacian is a self-adjoint operator with compact resolvent and hence its spectrum is purely discrete. This contrasts with the general case since, if (M, g) is a geodesically complete Riemannian manifold, not necessarily closed, it is known that its Laplacian $\Delta_g : C_c^\infty(M) \rightarrow L^2(M, dvol_g)$ is essentially self-adjoint but its spectral resolution is not purely discrete in general.

In order to show more clearly the new spectral phenomenons for Δ_g that the lost of compactness of M brings with, let us remind the spectral theorem for self-adjoint operators.

Theorem 1 [RS80] *Let A be a self-adjoint operator acting on a separable Hilbert space \mathcal{H} . Then, there exists a collection of measures on \mathbb{R} , $\{\mu_i\}_{i \in I}$ where $I \subseteq \mathbb{N}$, and there*

exists a unitary operator \mathcal{F} such that the following diagram commutes

$$\begin{array}{ccc} \mathcal{H} & \xrightarrow{A} & \mathcal{H} \\ \mathcal{F} \uparrow & & \downarrow \mathcal{F}^* \\ \bigoplus_{i \in I} L^2(\mathbb{R}, \mu_i) & \xrightarrow{\bigoplus_{i \in I} x_i} & \bigoplus_{i \in I} L^2(\mathbb{R}, \mu_i), \end{array}$$

where the operator $\bigoplus_{i \in I} x_i$ is the direct sum of the multiplication operators $x_i : L^2(\mathbb{R}, \mu_i) \rightarrow L^2(\mathbb{R}, \mu_i)$ that, specifically, send a function $f \in L^2(\mathbb{R}, \mu_i)$ to the function $x_i f$.

Given a self-adjoint operator A , as in the theorem above, the Lebesgue decomposition theorem for measures induces a decomposition of the Hilbert space \mathcal{H} into three important invariants subspaces \mathcal{H}_{ac} , \mathcal{H}_{pp} and \mathcal{H}_{sing} that, using theorem 1, are the associated to the three Hilbert spaces

$$\bigoplus_{\mu_i \text{ is ac}} L^2(\mathbb{R}, \mu_i), \quad \bigoplus_{\mu_i \text{ is pp}} L^2(\mathbb{R}, \mu_i) \text{ and } \bigoplus_{\mu_i \text{ is s}} L^2(\mathbb{R}, \mu_i),$$

where ac means absolutely continuous, pp means pure point and s means singular continuous measures (all of them defined respect to the Lebesgue measure in \mathbb{R}). We have already said that for (M, g) a closed Riemannian manifold, its Laplacian has purely discrete spectrum, in other words the invariant subspaces $L^2_{ac}(M, dvol_g)$ and $L^2_{sing}(M, dvol_g)$, corresponding to the self-adjoint operator $A := \Delta_g$, are empty. In this paper we will show examples of manifolds (M, g) for which $L^2_{ac}(M, dvol_g)$ is not empty and $L^2_{sing}(M, dvol_g)$ is empty.

The former description of the spectral analysis of self-adjoint operators justifies the question about how to prove that a geometric Laplacian does not have singular continuous spectrum. This paper provide, through the friendly environment of precise examples and without pretending to give a complete answer, an illustration of how to work around this question. The common feature of these examples is the control on the Riemannian metric of the open manifold at infinity.

Let us describe more carefully the contents. For Laplacians acting on manifolds with cylindrical or cusps ends and their natural generalizations it is well known in the literature (see [Gui89][Hus05] [Mül83]) how to find a meromorphic extension of their resolvent and how, using it, to prove the absence of singular continuous spectrum. In section 1 we illustrate the classical method to obtain a meromorphic extension of the resolvent for manifolds with cylindrical and cusp ends, and we show the relation of it with the absence of singular continuous spectrum, that is, fundamentally, the limit absorption principle. In section 2 we explain the notion of complete manifold with

corner of codimension 2 and how to prove absence of singular continuous spectrum for Laplacians on this kind of manifolds using the method of analytic dilation. Section 2 describe following [Can11] the way to apply the technique of analytic dilation to Laplacians on complete manifolds with corners of codimension 2, context on which the techniques explained in section 1 do not apply. The method of analytic dilation was originally applied to N -particle Schrödinger operators, a classic reference in that setting is [Gér93]. It has also been applied to the black-box perturbations of the Euclidean Laplacian in the series of papers [SZ93b] [SZ93a] [SZ94] [SZ95]. In [Bal97] it is used to study Laplacians on hyperbolic manifolds. The analytic dilation has also been applied to the study of the spectral and scattering theory of quantum wave guides and Dirichlet boundary domains, see e.g. [DEM98] [KS07]. It has also been applied to arbitrary symmetric spaces of noncompact type in [MV02] [MV04] [MV07]. In [Kal10] it is applied to manifolds with analytic asymptotically cylindrical end. In each of these settings new ideas and new methods carry out.

1 Meromorphic extension of the resolvent and singular continuous spectrum

In this section we give the main ideas of a method for meromorphically extend the resolvent of Laplacians on manifolds with cylindrical and cusps ends and show why such extension is enough to have absence of singular continuous spectrum. The results of this section were obtained in [Gui89][Mül83] but we base our exposition in [Hus05] since we consider it is easier to understand for non experts.

Definition 1 *Let M be an open manifold with a decomposition in a compact manifold M_0 with boundary $Y := \partial M_0$ and an open manifold M_∞ with boundary and suppose that $\partial M_\infty = M_0 \cap M_\infty = \partial M_0$. If M is endowed with a complete Riemannian metric such that*

i) M_∞ is isometric to $Y \times \mathbb{R}_+$ with the natural product metric $g_y + du \otimes du$, we say that M is a manifold with cylindrical end.

ii) M_∞ is isometric to $Y \times \mathbb{R}_+$ with the Riemannian metric $\frac{du \otimes du}{u^2} + \frac{g_y}{u^2}$, we say that M is a manifold with cusp end.

Let Δ_{cyl} be the Laplacian associated to a manifold (M, g_M) with cylindrical end. We recall that on $M_\infty = Y \times \mathbb{R}_+$ the Laplacian Δ_{cyl} has the form

$$\Delta_{0,M} := -\frac{\partial^2}{\partial u^2} + \Delta_Y$$

where Δ_Y is the Laplacian on Y associated to the Riemannian metric g_Y . Let (N^n, g_N) be a manifold with cusp, with $N_\infty = Y \times \mathbb{R}_+$, and let Δ_{cusp} be the Laplacian associated to (N^n, g_N) . Δ_{cusp} has the form

$$\Delta_{0,N} := -u^2 \frac{\partial^2}{\partial u^2} + (n-1)u \frac{\partial}{\partial u} + u^2 \Delta_Y$$

on N_∞ . An important approach, in quantum scattering theory, is to think that the Laplacians Δ_{cyl} and Δ_{cusp} are perturbations of the operators $\Delta_{0,M}$ and $\Delta_{0,N}$ respectively. Roughly speaking, this idea is formalized in the following way. Using Dirichlet boundary conditions, we consider the operators $\Delta_{0,M}$ and $\Delta_{0,N}$ as self-adjoint operators in $L^2(Y \times \mathbb{R}_+, dvol_{g_M})$ and $L^2(Y \times \mathbb{R}_+, dvol_{g_N})$. Observe that $L^2(Y \times \mathbb{R}_+, dvol_{g_M})$ is naturally isomorphic to $L^2(Y, dvol_{g_Y}) \otimes L^2(\mathbb{R}_+, du)$; furthermore, if $\phi_i \in C^\infty(Y)$ is an orthonormal basis of eigenvectors of Δ_Y with eigenvalues μ_i , then it provides an isomorphism of $L^2(Y, dvol_{g_Y}) \otimes L^2(\mathbb{R}_+, du)$ with $\oplus_{i \in \mathbb{N}} L^2(\mathbb{R}_+, du)$, hence $L^2(Y \times \mathbb{R}_+, dvol_{g_M}) \cong \oplus_{i \in \mathbb{N}} L^2(\mathbb{R}_+, du)$. Modulo this last isomorphism, we have:

$$\Delta_{0,M} = \bigoplus_{i \in \mathbb{N}} \left(-\frac{\partial^2}{\partial u^2} + \mu_i \right). \quad (1)$$

The analogue of (1) in the context of manifolds with cusps is

$$\Delta_{0,M} = \bigoplus_{i \in \mathbb{N}} \left(-u^2 \frac{\partial^2}{\partial u^2} + (n-1)u \frac{\partial}{\partial u} + u^2 \mu_j \right). \quad (2)$$

From this point we continue our exposition over the manifold with cylindrical end M , indicating how analogous methods apply in the case of manifolds with cusps and referring to [Mül83] for details.

Fourier transform and (2) give us a formula for the resolvent of the operators $-\frac{\partial^2}{\partial u^2} + \mu_i$ and hence for the resolvent of $\Delta_{0,M}$. The analogue of such a formula in the case of the manifold N with cusp is technically harder to obtain and can be found in [Mül83, lemma 2.68]. Define the double of the compact manifold with boundary M_0 as $\tilde{M} := M_0 \cup_Y M_0$, where we are identifying the boundary Y of two disjoint copies of M_0 and we endow \tilde{M} with the natural differential and Riemannian structure. We have also a nice formula

for the resolvent of the Laplacian $\Delta_{\tilde{M}}$ of \tilde{M} using a spectral resolution of $\Delta_{\tilde{M}}$. In order to apply this knowledge about the resolvents in M_∞ and \tilde{M} , we construct a *parametrix* for the resolvent of Δ_M i. e. an operator $P(\lambda)$ such that $R(\lambda) - P(\lambda)$ is compact in some weighted L^2 -space, where $\lambda \in \mathbb{C} - \mathbb{R}_+$ and $R(\lambda) := (\Delta_M - \lambda)^{-1}$ denotes the resolvent of Δ_M . We proceed as in [Hus05]: for $0 \geq a \geq b$, let $\rho(a, b) \in C^\infty(M, [0, 1])$ be such that

$$\rho(a, b)(x) = \begin{cases} 0 & \text{for } x \in M \cup (Y \times [0, a]); \\ 1 & \text{for } x \in Y \times [b, \infty). \end{cases}$$

We define the functions:

$$\begin{aligned} \Phi_1 &:= 1 - \rho\left(\frac{4}{5}, 1\right), & \Psi_1 &:= 1 - \rho\left(\frac{2}{5}, \frac{3}{5}\right) \\ \Phi_2 &:= \rho\left(0, \frac{1}{5}\right), & \Psi_2 &:= 1 - \Psi_1, \end{aligned}$$

for which $\Psi_1 + \Psi_2 = 1$,

$$\Phi_j(x) = 1 \text{ for } x \in \text{supp}(\Psi_j), \text{ and } \text{dist}(\text{supp}\nabla\Phi_j, \text{supp}\Psi_j) \geq \frac{1}{5}.$$

We define the operator $S(\lambda)$ with Schwartz kernel

$$S(x_1, x_2, \lambda) := \sum_{j=1}^2 \Psi_j(x_1) R_j(x_1, x_2, \lambda) \Phi_j(x_2), \quad (3)$$

where $R_1(x_1, x_2, \lambda)$ is the Schwartz kernel of the resolvent of the Laplacian $\Delta_{\tilde{M}}$, on \tilde{M} the double of the manifold with boundary M_0 , and $R_2(x_1, x_2, \lambda)$ is the Schwartz kernel of the resolvent of $\Delta_{0,M}$. Using the explicit expressions of the Schwartz kernels $R_1(x_1, x_2, \lambda)$ and $R_2(x_1, x_2, \lambda)$ and (1), it is possible to prove

Lemma 1 [Hus05, lemma 3.8] *For $\lambda \in \mathbb{C} - \mathbb{R}_+$, the operator $S(\lambda)$ is a parametrix of $R(\lambda)$ in the sense that $R(\lambda) - S(\lambda)$ is L^2 -compact.*

The meromorphic extension of the resolvent will have a domain contained in the following surface, that we call *spectral surface*:

$$\Sigma_s := \{\Lambda := (\Lambda_i) \in \mathbb{C}^{\mathbb{N}} : \forall i, j \in \mathbb{N}, \Lambda_i^2 + \mu_i = \Lambda_j^2 + \mu_j\}.$$

Σ_s is a covering of \mathbb{C} with projection $\pi_s(\Lambda) := \Lambda_i^2 + \mu_i$.

As we said previously, we will consider the resolvent acting on weighted L^2 -spaces that we shall define now:

$$L_\delta^2(M) := \{f : M \xrightarrow{\text{meas}} \mathbb{C} : \int_0^\infty \int_Y e^{2\delta u} |f(y, u)|^2 d\text{vol}_{g_Y} du < \infty\}. \quad (4)$$

For all $\delta > 0$, we have the inclusions

$$L_\delta^2(M) \subset L^2(M) \subset L_{-\delta}^2(M).$$

If we define the *physical domain* FD by

$$FD := \{\Lambda \in \Sigma_s : \Lambda_i \geq 0\},$$

then we can identify FD with $\mathbb{C} - \mathbb{R}_+$. We denote by Σ_s^μ the connected component of $\pi_s^{-1}(\mathbb{C} - [\mu, \infty))$ and, for $\epsilon > 0$, $\mu(\epsilon) := \min\{\mu \in \sigma(\Delta_Y) : \mu \leq \epsilon\}$. Now we can define the domains Ω_ϵ , for $\epsilon > 0$, where we will extend the resolvent of Δ_M ,

$$\Omega_\epsilon := (FD \cup \pi_s^{-1}(\{z \in \mathbb{C} : |z| \leq \epsilon\})) \cap \Sigma_s^{\mu(\epsilon)}$$

From the explicit formulas of the resolvents $R_1(\lambda)$ and $R_2(\lambda)$ in equation (3), and the definition of the weighed L^2 -spaces in (4), we deduce the following.

Lemma 2 [Hus05, lemma 3.20] *For all $\delta \geq \epsilon > 0$ the function $\lambda \mapsto S(\lambda)$ has a meromorphic extension to Ω_ϵ as a continuous operator from $L_\delta^2(M)$ to $L_{-\delta}^2(M)$.*

For $\Lambda \in \Omega_\epsilon$, let us define the operators

$$G(\Lambda) := S(\Lambda)(\Delta_M - \pi_s(\Lambda)) - Id.$$

We will use the following important tool of functional analysis to meromorphically extend the resolvent.

Theorem 2 [Hus05, appendix] (Analytic Fredholm theorem) *Let $U \subset \mathbb{C}$ be an open and connected set and let $T(z)$, for $z \in U$, be an analytic family of compact operators of a Hilbert space \mathcal{H} . Suppose that, for $z_0 \in U$, $(Id - T(z_0))^{-1}$ exists, then the family $(Id - T(z))^{-1}$ is meromorphic in U with values in the bounded linear operators of \mathcal{H} and poles contained in the set $\{z \in U : 1 \in \sigma(T(z))\}$.*

As for lemma 1, it is possible to prove that $G(\Lambda)$, as a bounded operator of $L_{-\delta}^2(M)$, is compact. This fact, lemma 2 and theorem 2 imply

Theorem 3 [Hus05, theorem 3.24] *The resolvent $R(\lambda)$ has a meromorphic extension from FD to Ω_ϵ as a continuous operator from $L^2_\delta(M)$ to $L^2_{-\delta}(M)$.*

The next theorem provides the connection between the meromorphic extension of the resolvent and the absence of singular continuous spectrum.

Theorem 4 [RS79] *Let H be a self-adjoint operator with resolvent $R(\lambda) := (H - \lambda)^{-1}$.*

i) Let (a, b) be a bounded interval and $\varphi \in \mathcal{H}$. Suppose that there exists $p > 1$ for which:

$$\sup_{0 < \epsilon < 1} \int_a^b |\operatorname{Im}(\varphi, R(x + i\epsilon)\varphi)|^p dx < \infty. \quad (5)$$

Then $E_{(a,b)}\varphi \in \mathcal{H}_{ac}$.

ii) Let (a, b) be a bounded interval. Suppose that there is a dense subset D in \mathcal{H} so that for $\varphi \in D$ the inequality (5) holds for some $p > 1$. Then H has purely absolutely continuous spectrum on (a, b) .

2 Analytic dilation on complete manifolds with corners of codimension 2

In [Mül96] it has been explained how to meromorphically extend the resolvent of a generalized Laplacian on a complete manifold with corner of codimension 2, using the method outlined in section 1 under the hypothesis that the Laplacian on the corner has kernel $\{0\}$. It turns out that to weaken this hypothesis and try to use the methods of section 1 is not easy and new methods should be used to prove absence of singular continuous spectrum. In this section we survey the method of analytic dilation applied to compatible Laplacians on complete manifolds with corners of codimension 2. This method appeared originally in the context of Shrödinger operators and was adapted in [Can11] to this geometric context.

Following [Mül96], we explain the notions of *compact and complete manifolds with corner of codimension 2*. Let X_0 be a compact oriented Riemannian manifold with boundary M and suppose that there exists a hypersurface Y of M that divides M in two manifolds with boundary M_1 and M_2 , i.e. $M = M_1 \cup M_2$ and $Y = M_1 \cap M_2$. Assume also that a neighborhood of Y in M is diffeomorphic to $Y \times (-\varepsilon, \varepsilon)$. We say that the manifold X_0 **has a corner of codimension 2** if X_0 is endowed with a Riemannian metric g that is a product metric on small neighborhoods, $M_i \times (-\varepsilon, 0]$ of the

M_i 's and on a small neighborhood, $Y \times (-\varepsilon, 0]^2$, of the corner Y . If X_0 has a corner of codimension 2, we say that X_0 is a **compact manifold with corner of codimension 2**.

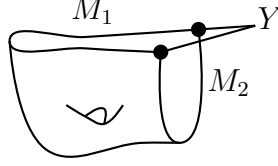


Figure 1. Compact manifold with corner of codimension 2.

Example 1 For $i = 1, 2$, let M_i be a compact oriented Riemannian manifold with boundary $\partial M_i := Y_i$. Suppose that on a neighborhood $Y_i \times (-\varepsilon, 0]$ of Y_i the Riemannian metric g_i of M_i is a product metric i.e. $g_i := g_{Y_i} + du \otimes du$ where u is the coordinate associated to $(-\varepsilon, 0]$ in $Y_i \times (-\varepsilon, 0]$. Then $M_1 \times M_2$ is a compact manifold with corner of codimension 2.

From the compact manifold with corner X_0 we construct a complete manifold X . Let

$$Z_i := M_i \cup_Y (\mathbb{R}^+ \times Y), \quad i=1,2,$$

where the bottom $\{0\} \times Y$ of the half-cylinder is identified with $\partial M_i = Y$. Then Z_i is a manifold with cylindrical end (see definition 1). Define the manifolds

$$W_1 := X_0 \cup_{M_2} (\mathbb{R}_+ \times M_2) \text{ and } W_2 := X_0 \cup_{M_1} (\mathbb{R}_+ \times M_1).$$

Observe that W_i is an n -dimensional manifold with boundary Z_i that can be equipped with a Riemannian metric compatible with the product metric of $\mathbb{R}_+ \times M_2$ and the Riemannian metric of X_0 . Set:

$$X := W_1 \cup_{Z_1} (\mathbb{R}_+ \times Z_1) = W_2 \cup_{Z_2} (\mathbb{R}_+ \times Z_2),$$

where we identify $\{0\} \times Z_i$ with Z_i , the boundary of W_i .

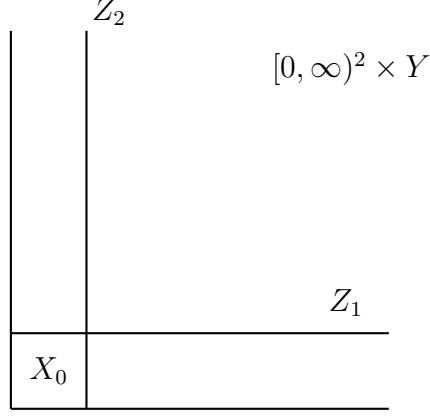


Figure 2. Sketch of a complete manifold with corner of codimension 2.

The above picture is an sketch, in particular the lines that enclosed the picture should not be thought as boundaries.

Let $T \geq 0$ be given and set

$$Z_{i,T} := M_i \cup_Y ([0, T] \times Y), \text{ for } i = 1, 2, \quad (6)$$

where $\{0\} \times Y$ is identified with Y , the boundary of M_i . $Z_{i,T}$ is a family of manifolds with boundary which exhausts Z_i . Next we attach to X_0 the manifold $[0, T] \times M_1$ by identifying $\{0\} \times M_1$ with M_1 . The resulting manifold $W_{2,T}$ is compact manifold with corner of codimension 2, whose boundary is the union of M_1 and $Z_{2,T}$. The manifold X has associated a natural exhaustion given by

$$X_T := W_{2,T} \cup_{Z_{2,T}} ([0, T] \times Z_{2,T}), \quad T \geq 0 \quad (7)$$

where we identify $Z_{2,T}$ with $\{0\} \times Z_{2,T}$.

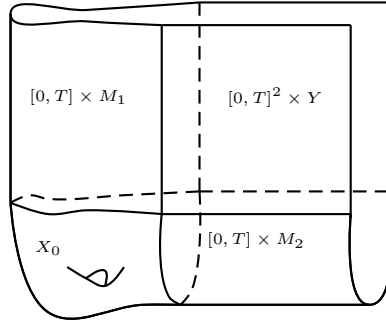


Figure 2. X_T , element of the exhaustion of X .

For each $T \in [0, \infty)$, X has two submanifolds with cylindrical ends, namely $M_i \times \{T\} \cup (Y \times \{T\}) \times [0, \infty)$, for $i = 1, 2$.

Let E be a Hermitian vector bundle over a complete manifold with corner of codimension 2, X . Let Δ be a generalized Laplacian acting on $C^\infty(X, E)$. The operator Δ is a **compatible Laplacian** over X if it satisfies the following properties:

- i) There exists a Hermitian vector bundle E_i over Z_i such that $E|_{\mathbb{R}_+ \times Z_i}$ is the pullback of E_i under the projection $\pi : \mathbb{R}_+ \times Z_i \rightarrow Z_i$, for $i = 1, 2$. We suppose also that the Hermitian metric of E is the pullback of the Hermitian metric of E_i . On $\mathbb{R}_+ \times Z_i$, we have

$$\Delta = -\frac{\partial^2}{\partial u_i^2} + \Delta_{Z_i},$$

where Δ_{Z_i} is a compatible Laplacian acting on $C^\infty(Z_i, E_i)$.

- ii) There exists a Hermitian vector bundle S over Y such that $E|_{\mathbb{R}_+^2 \times Y}$ is the pullback of S under the projection $\pi : \mathbb{R}_+^2 \times Y \rightarrow Y$. We assume also that the Hermitian product on $E|_{\mathbb{R}_+^2 \times Y}$ is the pullback of the Hermitian product on S . Finally we suppose that the operator Δ restricted to $\mathbb{R}_+^2 \times Y$ satisfies

$$\Delta = -\frac{\partial^2}{\partial u_1^2} - \frac{\partial^2}{\partial u_2^2} + \Delta_Y,$$

where Δ_Y is a generalized Laplacian acting on $C^\infty(Y, S)$.

Examples of compatible Laplacians are the Laplacian acting on forms and Laplacians associated to compatible Dirac operators (see [Mül96]). Since X is a manifold with bounded geometry and the vector bundle E has bounded Hermitian metric, the operator $\Delta : C_c^\infty(X, E) \rightarrow L^2(X, E)$ is essentially self-adjoint (see [Shu91, corollary 4.2]). Similarly $\Delta_{Z_i} : C_c^\infty(Z_i, E_i) \rightarrow L^2(Z_i, E_i)$ is also essentially self-adjoint for $i = 1, 2$.

Definition 2 • Let H and $H^{(i)}$ be the self-adjoint extensions of $\Delta : C_c^\infty(X, E) \rightarrow L^2(X, E)$ and $\Delta_{Z_i} : C_c^\infty(Z_i, E_i) \rightarrow L^2(Z_i, E_i)$.

- Let b_i be the self-adjoint extension of $-\frac{d^2}{du_i^2} : C_c^\infty(\mathbb{R}_+) \rightarrow L^2(\mathbb{R}_+)$ obtained by imposing Dirichlet boundary conditions at 0.
- Let H_i be the self-adjoint operator $b_i \otimes Id + Id \otimes H^{(i)}$ acting on $L^2(\mathbb{R}_+) \otimes L^2(Z_i, E_i)$.
- Let $H^{(3)}$ be the self-adjoint operator associated to the essentially self-adjoint operator $\Delta_Y : C^\infty(Y, S) \rightarrow L^2(Y, S)$ and let H_3 be the self-adjoint operator $H_3 := b_1 \otimes Id \otimes Id + Id \otimes b_2 \otimes Id + Id \otimes Id \otimes H^{(3)}$ acting on $L^2(\mathbb{R}_+) \otimes L^2(\mathbb{R}_+) \otimes L^2(Y)$.
- The operators H_i are called **channel operators** for $i = 1, 2, 3$.

The self-adjoint operators H_1 and H_2 have a free channel of dimension 1 (associated to b_1 and b_2 , respectively); the operator H_3 has a free channel of dimension 2 (associated to $b_1 \otimes Id \otimes Id + Id \otimes b_2 \otimes Id$). In some parts of this text we abuse the notation and denote by H , H_i , and $H^{(i)}$ the Laplacians acting on distributions and the self-adjoint operators previously defined.

Along the next section we will use the following notation. Let H be a self-adjoint operator acting on a Hilbert space \mathcal{H} . We define the Banach space $\mathcal{H}_2(H)$ as the domain of H with the norm $\|\varphi\|_2 := \|(|H| + i)\varphi\|$. Similarly, we define the Banach spaces $\mathcal{H}_1(H)$ as the completion of $\mathcal{H}_2(H)$ with the norm $\|\varphi\|_1 := \|(|H| + i)^{1/2}\varphi\|$, and $\mathcal{H}_{-1}(H)$ and $\mathcal{H}_{-2}(H)$ the dual spaces associated to $\mathcal{H}_1(H)$ and $\mathcal{H}_2(H)$.

2.1 Analytic dilation

The analytic dilation of a many-body Schrödinger operator depends on the analytic dilation of their subsystem Hamiltonians (see [?]). In a similar way the analytic dilation of H is described in terms of the spectral theory of the operators $H^{(1)}$, $H^{(2)}$ and $H^{(3)}$, explained above. For $\theta > 0$, we define the operator $U_{i,\theta} : L^2(Z_i) \rightarrow L^2(Z_i)$ that essentially is the dilation operator by $\theta + 1$ up to a compact set. More precisely:

$$U_{i,\theta}f(x) = \begin{cases} f(x) & \text{for } x \in M_i. \\ (\theta + 1)^{1/2}f((\theta + 1)u, y) & \text{for } x = (u, y) \in [0, \infty) \times Y \\ & \text{and for } u \text{ big enough,} \end{cases}$$

and $U_{i,\theta}f$ is extended to the whole manifold Z_i in such a way that it sends $C_c^\infty(Z_i)$ into $C_c^\infty(Z_i)$, and it becomes a unitary operator on $L^2(Z_i)$. We refer to [Can11] for the technical details. Similarly, the operators $U_\theta : L^2(X) \rightarrow L^2(X)$ are defined by

$$U_\theta f(x) = \begin{cases} f(x) & \text{for } x \in X_0. \\ (\theta + 1)^{1/2}U_{i,\theta}f((\theta + 1)u_i, z_i) & \text{for } x = (u_i, z_i) \in [0, \infty) \times Z_i \\ & \text{and for } u_i \text{ big enough.} \end{cases}$$

Again $U_\theta f$ is extended to the whole X in such a way that, for $f \in C_c^\infty(X)$, $U_\theta f \in C_c^\infty(X)$, and U_θ becomes a unitary operator in $L^2(X)$.

For $\theta \in [0, \infty)$, define $H_\theta := U_\theta H U_\theta^{-1}$, a closed operator with domain $\mathcal{H}_2(H)$. We

have that

$$\mathcal{H}_2(H) = \{f \in L^2(X) : \Delta_{dist} f \in L^2(X)\}$$

is the second Sobolev space associated to (X, g) . Consider the set:

$$\Gamma := \{\theta := \theta_0 + i\theta_1 \in \mathbb{C} : \theta_0 > 0, \theta_0 > |\theta_1| \text{ and } \operatorname{Im}(\theta)^2 < 1/2\}. \quad (8)$$

We will extend the family H_θ from $[0, \infty)$ to Γ .

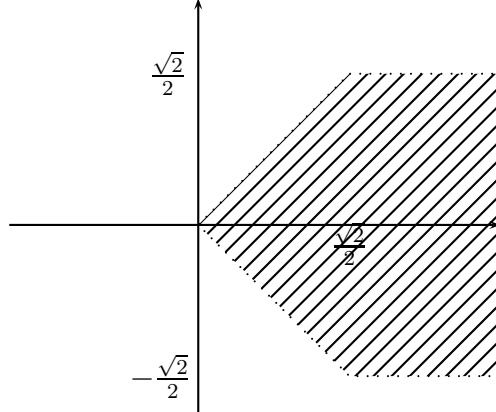


Figure 4. Sketch of the region Γ .

In [Can11], the next theorem is proved:

Theorem 5 [Can11] *The family $(H_\theta)_{\theta \in [0, \infty)}$ extends to an holomorphic family for $\theta \in \Gamma$, which satisfies:*

1) H_θ is a closed operator with domain $\mathcal{H}_2(H)$ for $\theta \in \Gamma$.

2) For $\varphi \in \mathcal{H}_2(H)$ the map $\theta \mapsto H_\theta \varphi$ is holomorphic in Γ .

An holomorphic family of operators satisfying 1) and 2) is called a **holomorphic family of type A**. This theorem is proved using the analogous result that the family $\{H^{(i), \theta}\}_{\theta \in [0, \infty)}$ extends to a holomorphic family of type A in Γ , where $H^{(i), \theta}$ denotes the closed operator associated to $U_{i, \theta} \Delta_{Z_i} U_{i, \theta}^{-1}$ with domain

$$\mathcal{H}_2(H^i) = \{f \in L^2(Z_i) : \Delta_{dist}(f) \in L^2(Z_i)\},$$

the second Sobolev space associated to (Z_i, g_i) .

The families H_θ and $H_\theta^{(i)}$ extend to domains larger than Γ , but Γ is enough for our goals. In particular, Γ is chose because for $\theta \in \Gamma$ is easy to prove that $H_\theta^{(i)}$ is m -sectorial

(see [Can11, section 2.7]), fact that will be important for the proof of theorem 6 where Ichinose lemma is a main tool. We define

$$\theta' := \frac{1}{(\theta + 1)^2}.$$

The parameter θ' is very important in the description of the essential spectrum of H_θ as we can see in the next theorem.

Theorem 6 [Can11] *For $\theta \in \Gamma$,*

$$\begin{aligned} \sigma_{ess}(H_\theta) = & \bigcup_{\mu \in \sigma(H^{(3)})} (\mu + \theta'[0, \infty)) \\ & \cup \bigcup_{\lambda_1 \in \sigma_{pp}(H^{(1), \theta})} (\lambda_1 + \theta'[0, \infty)) \\ & \cup \bigcup_{\lambda_2 \in \sigma_{pp}(H^{(2), \theta})} (\lambda_2 + \theta'[0, \infty)). \end{aligned}$$

It is possible to associate to $(U_\theta)_{\theta \in [0, \infty)}$ a set of function $\mathcal{V} \subset \mathcal{H}_2(H)$ that satisfies:

- i) \mathcal{V} is dense in $L^2(X)$.
- ii) for $\varphi \in \mathcal{V}$, $U_\theta \varphi$ is defined for all $\theta \in \Gamma$.
- iii) $U_\theta \mathcal{V}$ is dense in $L^2(X)$ for all $\theta \in \Gamma$.

The elements of a subset of $\mathcal{H}_2(H)$ which satisfies i) and ii) will be called **analytic vectors**. We denote by Λ the left-hand plane, more explicitly:

$$\Lambda := \{(x, y) \in \mathbb{C} : x < 0\}. \quad (9)$$

We denote by $R(\lambda)$ the resolvent of H and by $R(\lambda, \theta)$ the resolvent of H_θ . Using the general analytic dilation theory of Aguilar-Balslev-Combes (see [Bal97]) we describe the nature of the spectrum of H . This theory is based on:

- i) The knowledge of the essential spectrum of H_θ , provided by theorem 6.
- ii) The following equation, that is consequence of the unitarity of U_θ ,

$$\langle R(\lambda)f, g \rangle_{L^2(X)} = \langle R(\lambda, \theta)U_\theta f, U_\theta g \rangle_{L^2(X)}, \quad (10)$$

for $f, g \in \mathcal{V}$ and $\theta \in [0, \infty)$.

Since the right-hand side of (10) is defined for $\lambda \in \Lambda$ and $\theta \in \Gamma$, (10) provides a meromorphic extension of the functions $\lambda \mapsto \langle R(\lambda)f, g \rangle_{L^2(X)}$ from Λ to $\mathbb{C} - \sigma(H_\theta)$. From this, using Aguilar-Balslev-Combes, we deduce the following theorem.

Theorem 7 [Can11] 1) For $f, g \in \mathcal{A}$ the function $\lambda \mapsto \langle R(\lambda)f, g \rangle_{L^2(X)}$ extends from Λ to $\mathbb{C} - \sigma(H_\theta)$.

2) For all $\theta \in \Gamma$, H_θ has no singular continuous spectrum.

3) The accumulation points of $\sigma_{pp}(H)$ are contained in $\{\infty\} \cup \sigma(H^{(3)}) \cup \bigcup_{i=1}^2 \sigma_{pp}(H^{(i)})$.

In the case of manifolds with cylindrical ends we have shown in section 1 the absence of singular continuous spectrum for their Laplacians; in [Don84], by giving a polynomial bound to the growing of the number of L^2 -eigenvalues, it is proved that the unique possible accumulation point of the pure point spectrum of a Laplacian on a manifold with cylindrical end is ∞ .

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